Characterization of several saddle point concepts for set-valued maps(Nonlinear Analysis and Convex Analysis)

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Characterization of several saddle point concepts for set-valued maps*

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Abstract: Based on the relationship between two sets with respect to a convex cone, we introduce six different solution concepts on set-valued optimization problems. By using a nonlinear scalarization method, we obtain optimality conditions for efficient solutions of set-valued optimization problems. Moreover, we observe several kinds of concepts for saddle points of set-valued maps including an idea of efficient-like saddle point.

Key words: Nonlinear scalarization, vector optimization, set-valued optimization, saddle point, efficient solution, optimality conditions.

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1 Introduction

The notion of saddle point for set-valued maps was introduced by Luc and Vargas [8] in 1992, and also a loose saddle point theorem for set-valued maps was established. In 1996, Tan et al. [13] presented loose saddle point theorems in topological vector spaces (not locally convex) which were proved by a way to be different from those of Luc and Vargas, and were weakened with respect to the continuity of set-valued map. By defining

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a loose semisaddle point and using a lemma related to it, a loose saddle point theorem which is based on continuity and quasiconvexity–quasiconcavity of its scalarized maps was established by Kim et al. [4] in 1999. Moreover, generalized loose saddle point theorems were showed by Lin [6] in 1999.

On the one hand, in recent study on set-valued optimization problems, some solution concepts are defined by the efficiency of vectors as elements of set-valued objective functions based on a preorder which is a comparison between vectors with respect to a convex cone; see, [9] and [14]. Generally, an optimization problem with a set-valued objective function is very interesting from the point of view practically as well as theoretically, but there are many possibilities to chose a suitable criterion to optimize feasible sets. Mathematical methodology on the comparison between sets is not so popular, and hence we study characterizations of set-valued maps via scalarization and we observe optimality conditions for efficient solutions of set-valued optimization problems and several kinds of concepts for saddle points of set-valued maps.

When we consider a vector optimization problem, we use some kinds of scalarization methods to get an equivalent scalar problem, and then we get an optimal solution and its value for the scalar problem much easier because the target space is one dimensional space and it is a total ordering space. Georgiev and Tanaka [1, 2] generalized Fan's inequality for set-valued maps by using a nonlinear scalarizing function regarded as a generalization of the Tchebyshev scalarization, which is well known and one of scalarization methods overcoming some nonconvexity in vector optimization. This kind of scalarizing function inherits some types of cone-convexity and cone-continuity from the parent set-valued map. The study on this kind of scalarizing function is also found in some other papers (e.g., [3]), and it is referred to as the smallest strictly monotonic function. Based on this approach, Nishizawa et al [10] have researched such inherited properties of the scalarizing function.

The aim of this paper is to investigate how set-valued maps are characterized via scalarization and to give optimality conditions for efficient solutions of set-valued optimization problems. Moreover, we observe several kinds of concepts for saddle points of set-valued maps including an idea of efficient-like saddle point.

Concretely speaking, the organization of the paper is as follows. In Section 2, based on the comparisons between two sets used in [5], we introduce six different solution concepts on the same problem but by defining six types of efficiency on images of set-valued objective functions directly. In Section 3, we introduce a nonlinear scalarization method, which involves a sublinear function $h_C(y; k) := \inf\{t : y \in tk - C\}$ where $C \neq Y$ is a convex cone with nonempty interior in a real topological vector space Y and $k \in \operatorname{int} C$. In Section 4, by using the nonlinear scalarization method, we obtain optimality conditions for efficient solutions of set-valued optimization problems.

2 Relationships between Two Sets

In this section, we introduce relationships between two sets in a vector space. Throughout the paper, let Z be a real ordered topological vector space with the vector ordering $\leq_C$ induced by a convex cone C: for $x, y \in Z$,

$$x \leq_C y \text{ if } y - x \in C.$$

First, we consider comparison methodology between two vectors with respect to C. There are two types of comparable cases and one in-comparable case. Comparable cases are as follows: for $a, b \in Z$,

$$(1)\ a \in b - C \text{ (i.e., } a \leq_C b), \qquad (2)\ a \in b + C \text{ (i.e., } b \leq_C a).$$

When we replace a vector $a \in Z$ with a set $A \subset Z$, that is, we consider a comparison between a set and a vector, there are four types of completely comparable and partially comparable cases and one in-comparable case. Such comparable cases are as follows: for $A \subset Z$ and $b \in Z$,

$$(1)\ A \subset (b - C), \qquad (2)\ A \cap (b - C) \neq \phi,$$
$$(3)\ A \cap (b + C) \neq \phi, \qquad (4)\ A \subset (b + C).$$

By the same way, when we replace a vector $a \in Z$ with a set $A \subset Z$, that is, we consider comparison between two sets with respect to C, there are twelve types of some what comparable cases and in-comparable case. For two sets $A, B \subset Z$, A would be inferior to B if we have one of the following situations:

$$(1)\ A \subset (\cap_{b \in B}(b - C)), \qquad (2)\ A \cap (\cap_{b \in B}(b - C) \neq \phi,$$
$$(3)\ (\cup_{a \in A}(a + C)) \supset B, \qquad (4)\ (\cup_{a \in A}(a + C)) \cup B,$$
$$(5)\ (\cap_{a \in A}(a + C)) \supset B, \qquad (6)\ ((\cap_{a \in A}(a + C)) \cap B) \neq \phi,$$
$$(7)\ A \subset (\cup_{b \in B}(b - C)), \qquad (8)\ (A \cap (\cup_{b \in B}(b - C)) \neq \phi).$$

Also, there are eight converse situations in which B would be inferior to A. Actually the relationships (1) and (4) in the above comparison of A and B coincide with the relationships (5) and (8), respectively. Therefore, we define the following six kinds of classification for set-relationships.

Definition 2.1. (Set-relationships in [5]) Given nonempty sets $A, B \subset Z$, we define six types of relationships between A and B as follows:

$$(1)\ A \leq_C^{(1)} B \text{ by } A \subset \cap_{b \in B}(b - C), \qquad (2)\ A \leq_C^{(2)} B \text{ by } A \cap (\cap_{b \in B}(b - C)) \neq \phi,$$
$$(3)\ A \leq_C^{(3)} B \text{ by } \cup_{a \in A}(a + C) \supset B, \qquad (4)\ A \leq_C^{(4)} B \text{ by } (\cap_{a \in A}(a + C)) \cap B \neq \phi,$$
$$(5)\ A \leq_C^{(5)} B \text{ by } A \subset \cup_{b \in B}(b - C), \qquad (6)\ A \leq_C^{(6)} B \text{ by } A \cap (\cup_{b \in B}(b - C)) \neq \phi.$$

Proposition 2.1. **([5])** *For nonempty sets $A, B \in Z$ and a convex cone C in Z, the following statements hold:*

$$A \leq_C^{(1)} B \textit{ implies } A \leq_C^{(2)} B; \qquad A \leq_C^{(1)} B \textit{ implies } A \leq_C^{(4)} B;$$

$$A \le_C^{(2)} B \text{ implies } A \le_C^{(3)} B; \qquad A \le_C^{(4)} B \text{ implies } A \le_C^{(5)} B;$$
$$A \le_C^{(3)} B \text{ implies } A \le_C^{(6)} B; \qquad A \le_C^{(5)} B \text{ implies } A \le_C^{(6)} B.$$

3 Nonlinear Scalarization Method

We introduce a nonlinear scalarization for set-valued maps and show some properties on a characteristic function and scalarizing functions introduced in the sequel.

Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. We assume that $C \neq Y$, which is equivalent to

$$\operatorname{int} C \cap (-\operatorname{cl} C) = \emptyset \tag{3.1}$$

for a convex cone with nonempty interior, where $\operatorname{int} C$ and $\operatorname{cl} C$ denote the interior and the closure of C, respectively.

To begin with, we define a characteristic function

$$h_C(y; k) := \inf\{t : y \in tk - C\}$$

where $k \in \operatorname{int} C$ and moreover we get $-h_C(-y; k) = \sup\{t : y \in tk + C\}$. This function $h_C(y; k)$ has been treated in some papers; e.g., see [3] and [12], and it is regarded as a generalization of the Tchebyshev scalarization. Essentially, $h_C(y; k)$ is equivalent to the smallest strictly monotonic function with respect to $\operatorname{int} C$ defined by Luc in [7]. Note that $h_C(\cdot; k)$ is positively homogeneous and subadditive for every fixed $k \in \operatorname{int} C$, and hence it is sublinear and continuous.

Now, we give some useful properties of this function h_C, which are proved in [9].

Lemma 3.1. *Let $y \in Y$, then the following statements hold:*

(i) *If $y \in -\operatorname{int} C$, then $h_C(y; k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $h_C(y; k) < 0$, then $y \in -\operatorname{int} C$.*

Proof. First we prove the statement (i). Suppose that $y \in -\operatorname{int} C$, then there exists an absorbing neighborhood V_0 of 0 in Y such that $y + V_0 \subset -\operatorname{int} C$. Since V_0 is absorbing, for all $k \in \operatorname{int} C$, there exists $t_0 > 0$ such that $t_0 k \in V_0$. Therefore, $y + t_0 k \in y + V_0 \subset -\operatorname{int} C$. Thus, we have $y \in -t_0 k - \operatorname{int} C \subset -t_0 k - C$. Hence, $\inf\{t : y \in tk - C\} \le -t_0 < 0$, which shows that $h_C(y; k) < 0$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \operatorname{int} C$ such that $h_C(y; k) < 0$. Then, there exist $t_0 > 0$ and $c_0 \in C$ such that $y = -t_0 k - c_0 = -(t_0 k + c_0)$. Since $t_0 k \in \operatorname{int} C$ and C is a convex cone, we have $y \in -\operatorname{int} C$. □

Lemma 3.2. *Let $y \in Y$, then the following statements hold:*

(i) *If* $y \in -\mathrm{cl}\, C$, *then* $h_C(y;k) \leq 0$ *for all* $k \in \mathrm{int}\, C$;

(ii) *If there exists* $k \in \mathrm{int}\, C$ *with* $h_C(y;k) \leq 0$, *then* $y \in -\mathrm{cl}\, C$.

Proof. First we prove the statement (i). Suppose that $y \in -\mathrm{cl}\, C$. Then, there exists a net $\{y_\lambda\} \subset -C$ such that y_λ converges to y. For each y_λ, since $y_\lambda \in 0 \cdot k - C$ for all $k \in \mathrm{int}\, C$, $h_C(y_\lambda;k) \leq 0$ for all $k \in \mathrm{int}\, C$. By the continuity of $h_C(\cdot;k)$, $h_C(y;k) \leq 0$ for all $k \in \mathrm{int}\, C$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \mathrm{int}\, C$ such that $h_C(y;k) \leq 0$. In the case $h_C(y;k) < 0$, from (ii) of Lemma 3.1, it is clear that $y \in -\mathrm{cl}\, C$. Then we assume that $h_C(y;k) = 0$ and show that $y \in -\mathrm{cl}\, C$. By the definition of h_C, for each $n = 1, 2, \ldots$, there exists $t_n \in R$ such that

$$h_C(y;k) \leq t_n < h_C(y;k) + \frac{1}{n} \tag{3.2}$$

and

$$y \in t_n k - C. \tag{3.3}$$

Condition (3.2) implies $\lim_{n\to\infty} t_n = 0$. Hence by condition (3.3), there exists $c_n \in C$ such that $y = t_n k - c_n$, that is, $c_n = t_n k - y$. Since $c_n \to -y$ as $n \to \infty$, we have $y \in -\mathrm{cl}\, C$. $\square$

Lemma 3.3. *Let* $y, \bar{y} \in Y$, *then the following statements hold:*

(i) *If* $y \in \bar{y} + \mathrm{int}\, C$, *then* $h_C(y;k) > h_C(\bar{y};k)$ *for all* $k \in \mathrm{int}\, C$;

(ii) *If* $y \in \bar{y} + \mathrm{cl}\, C$, *then* $h_C(y;k) \geq h_C(\bar{y};k)$ *for all* $k \in \mathrm{int}\, C$.

Lemma 3.4. *Let* $y, \bar{y} \in Y$ *and* $\bar{y} \in \mathrm{int}\, C$, *then the following statements hold:*

(i) *If* $\bar{y} \notin y + \int C$, *then* $h_C(\bar{y};\bar{y}) \leq h_C(y;\bar{y})$;

(ii) *If* $\bar{y} \notin y + \mathrm{cl}\, C$, *then* $h_C(\bar{y};\bar{y}) < h_C(y;\bar{y})$.

Lemma 3.5. *Let* $y, \bar{y} \in Y$, *then the following statements hold:*

(i) *If* $\bar{y} \notin y + \mathrm{int}\, C$, *then* $-h_C(-\bar{y};k) \leq h_C(y;k)$ *for all* $k \in \mathrm{int}\, C$;

(ii) *If* $\bar{y} \notin y + \mathrm{cl}\, C$, *then* $-h_C(-\bar{y};k) < h_C(y;k)$ *for all* $k \in \mathrm{int}\, C$.

Lemma 3.6. *Let* $y, \bar{y} \in Y$ *and* $\bar{y} \in \mathrm{int}\, C$, *then the following statements hold:*

(i) *If* $\bar{y} \in y + \mathrm{int}\, C$, *then* $h_C(y;\bar{y}) < -h_C(-\bar{y};\bar{y})$;

(ii) *If* $\bar{y} \in y + \mathrm{cl}\, C$, *then* $h_C(y;\bar{y}) \leq -h_C(-\bar{y};\bar{y})$.

Remark 3.1. In the above lemma, we note that each converse does not hold.

Now, we consider several characterizations for images of a set-valued map by the nonlinear and strictly monotone characteristic function h_C. We observe the following four types of scalarizing functions:

$$\begin{aligned}
&(1)\quad \psi_C^F(x;k) := \sup\{h_C(y;k) : y \in F(x)\}, \\
&(2)\quad \varphi_C^F(x;k) := \inf\{h_C(y;k) : y \in F(x)\}, \\
&(3)\quad -\varphi_C^{-F}(x;k) = \sup\{-h_C(-y;k) : y \in F(x)\}, \\
&(4)\quad -\psi_C^{-F}(x;k) = \inf\{-h_C(-y;k) : y \in F(x)\}.
\end{aligned}$$

Functions (1) and (4) have symmetric properties and then results for function (4) $-\psi_C^{-F}$ can be easily proved by those for function (1) ψ_C^F. Similarly, the results for function (3) $-\varphi_C^{-F}$ can be deduced by those for function (2) φ_C^F. By using these four functions we measure each image of set-valued map F with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image with respect to convex cone C.

Without proofs, which are referred in [9], we state the following characterizations on the scalarizing functions above.

Proposition 3.1. *Let $x \in X$, then the following statements hold:*

(i) *If $F(x) \cap (-\operatorname{int} C) \neq \emptyset$, then $\varphi_C^F(x;k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $\varphi_C^F(x;k) < 0$, then $F(x) \cap (-\operatorname{int} C) \neq \emptyset$.*

Proposition 3.2. *Let $x \in X$, then the following statements hold:*

(i) *If $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$, then $\varphi_C^F(x;k) \leq 0$ for all $k \in \operatorname{int} C$;*

(ii) *If $F(x)$ is a compact set and there exists $k \in \operatorname{int} C$ with $\varphi_C^F(x;k) \leq 0$, then $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$.*

4 Optimality Conditions for Set-Valued Optimization Problems

In this section, we introduce new definitions of efficient solution for set-valued optimization problems. Using the sclarization method introduced in Section 3, we obtain optimal sufficient conditions on such efficiency. Throughout the section, let X be a nonempty set, Y a real ordered topological vector space with convex cone C. We assume that $C \neq Y$ and $\operatorname{int} C \neq \emptyset$. Let $F : X \to 2^Y$ be a set-valued map. A set-valued optimization problem is written as

(SVOP) min F(x) subject to $x \in V$, where $V = \{x \in X : F(x) \neq \phi\}$.

In this problem, we were defined an efficient solution as follows ever. Vector $x_0 \in V$ is an efficient solution of (SVOP) if there exists $y_0 \in F(x_0)$ such that $(F(x)\backslash\{y_0\}) \cap (y_0 - C) = \phi$ for all $x \in V$. This type of solution is defined based on a comparison between vectors. However F is a set-valued map, so it is natural to define efficient solution concepts based on direct comparisons between sets given in Definition 2.1.

Definition 4.1. (Efficient solution of (SVOP)) $x_0 \in V$ is said to be an efficient (resp., strongly efficient) solution for (SVOP) with respect to $\leq_C^{(i)}$for $i = 1, \ldots, 6$ if there exists no $x \in V\backslash\{x_0\}$ satisfying $F(x) \leq_C^{(i)} F(x_0)$ (resp., $F(x) \leq_{\mathrm{cl}\,C}^{(i)} F(x_0)$) for $i = 1, \ldots, 6$, respectively.

Using sclarization functions introduced in Section 3, we obtain the following optimality conditions for (SVOP).

Theorem 4.1. *Let $x_0 \in V$ and $F(x_0) \subset \mathrm{int}\, C$. x_0 is a strongly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{cl}\,C}^{(1)}$ if and only if for any $x \in V\backslash\{x_0\}$, there exists $k \in \mathrm{int}\, C$ such that $\varphi_C^F(x_0; k) < \psi_C^F(x; k)$.*

Proof. Suppose that for any $x \in V\backslash\{x_0\}$, there exists $k \in \mathrm{int}\, C$ such that $\varphi_C^F(x_0; k) < \psi_C^F(x; k)$. Assume that x_0 in not a strongly efficient solution with respect to $\leq_{\mathrm{cl}\,C}^{(1)}$. Then there exists $\bar{x} \in V\backslash\{x_0\}$ such that $F(\bar{x}) \leq_{\mathrm{cl}\,C}^{(1)} F(x_0)$. From the condition (ii) in Lemma 3.3, it follows that $h_C(\bar{y}; k) \leq h_C(y_0; k)$ for any $k \in \mathrm{int}\, C$. Hence we get $\psi_C^F(\bar{x}; k) \leq \varphi_C^F(x_0; k)$, which contradicts to the assumption.

On the other hand, suppose that x_0 is a strongly efficient solution with respect to $\leq_{\mathrm{cl}\,C}^{(1)}$. Then there no exists $x \in V\backslash\{x_0\}$ such that $F(x) \leq_{\mathrm{cl}\,C}^{(1)} F(x_0)$. Hence for any $x \in V\backslash\{x_0\}$, there exist $\bar{y} \in F(x)$ and $\bar{y_0} \in F(x_0)$ such that $\bar{y} \notin \bar{y_0} - \mathrm{cl}\, C$. From the condition (ii) in Lemma 3.4, it follows that $h_C(\bar{y_0}; \bar{y_0}) < h_C(\bar{y}; \bar{y_0})$, and hence $\varphi_C^F(x_0; \bar{y_0}) < \psi_C^F(x; \bar{y_0})$. □

Theorem 4.2. *Let $x_0 \in V$. Suppose $F(x_0) \subset \mathrm{int}\, C$ and $F(x)$ is compact for all $x \in V$. If x_0 is a strongly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{cl}\,C}^{(2)}$ then $-\psi_C^{-F}(x_0; k) < \varphi_C^F(x; k)$ for any $k \in \mathrm{int}\, C$ and $x \in V\backslash\{x_0\}$.*

Proof. Suppose that x_0 is a strongly efficient solution with respect to $\leq_{\mathrm{cl}\,C}^{(2)}$. Then there no exists $x \in V\backslash\{x_0\}$ such that $F(x) \leq_{\mathrm{cl}\,C}^{(2)} F(x_0)$. Hence for any $x \in V\backslash\{x_0\}$, there exists $\hat{y_0} \in F(x_0)$ such that $y \notin \hat{y_0} - \mathrm{cl}\, C$ for any $y \in F(x)$. From the condition (ii) in Lemma 3.5, it follows that $-h_C(-\hat{y_0}; k) < h_C(y; k)$ for any $k \in \mathrm{int}\, C$. From the compactness of $F(x)$, there exists $\hat{y} \in F(x)$ such that $h_C(\hat{y}; k) = \inf_{y \in F(x)}\{h_C(y; k)\}$, and hence $-\psi_C^{-F}(x_0; k) < \varphi_C^F(x; k)$. □

Theorem 4.3. *Let $x_0 \in V$. Suppose $F(x_0) \subset \mathrm{int}\, C$ and $F(x)$ is compact for all $x \in V$. x_0 is a strongly efficient solution for* (SVOP) *with respect to $\leq_{\mathrm{cl}\,C}^{(3)}$ if and only if for any $x \in V\backslash\{x_0\}$, there exists $k \in \mathrm{int}\, C$ such that $\varphi_C^F(x_0; k) < \varphi_C^F(x; k)$.*

Proof. Suppose that for any $x \in V\backslash\{x_0\}$, there exists $k \in \operatorname{int} C$ such that $\varphi_C^F(x_0;k) < \varphi_C^F(x;k)$. Assume that x_0 in not a strongly efficient solution with respect to $\leq_{\operatorname{cl} C}^{(3)}$. Then there exists $\bar{x} \in V\backslash\{x_0\}$ such that $F(\bar{x}) \leq_{\operatorname{cl} C}^{(3)} F(x_0)$. From the condition (ii) in Lemma 3.3, it follows that $h_C(\hat{y};k) \leq h_C(y_0;k)$ for any $k \in \operatorname{int} C$. Hence we get $\varphi_C^F(\bar{x};k) \leq \varphi_C^F(x_0;k)$, which contradicts to the assumption.

On the other hand, suppose that x_0 is a strongly efficient solution with respect to $\leq_{\operatorname{cl} C}^{(3)}$. Then there no exists $x \in V\backslash\{x_0\}$ such that $F(x) \leq_{\operatorname{cl} C}^{(3)} F(x_0)$. Hence for any $x \in V\backslash\{x_0\}$, there exists $\bar{y_0} \in F(x_0)$ such that $\bar{y_0} \notin y + \operatorname{cl} C$ for any $y \in F(x)$. From the condition (ii) in Lemma 3.4, it follows that $h_C(\bar{y_0};\bar{y_0}) < h_C(y;\bar{y_0})$. From the compactness of $F(x)$, there exists $\hat{y} \in F(x)$ such that $h_C(\hat{y};\bar{y_0}) = \inf_{y \in F(x)}\{h_C(y;y_0)\}$. Hence we have $\varphi_C^F(x_0;\bar{y_0}) < \varphi_C^F(x;\bar{y_0})$. □

Theorem 4.4. *Let $x_0 \in V$. Suppose $F(x_0) \subset \operatorname{int} C$ and $F(x)$ is compact for all $x \in V$. x_0 is a strongly efficient solution for* (SVOP) *with respect to $\leq_{\operatorname{cl} C}^{(4)}$ if and only if $-\varphi_C^{-F}(x_0;k) < \psi_C^F(x;k)$ for any $x \in V\backslash\{x_0\}$ and $k \in \operatorname{int} C$.*

Proof. Suppose that $-\varphi_C^{-F}(x_0;k) < \psi_C^F(x;k)$ for any $x \in V\backslash\{x_0\}$ and $k \in \operatorname{int} C$. Assume that x_0 in not a strongly efficient solution with respect to $\leq_{\operatorname{cl} C}^{(4)}$. Then there exists $\bar{x} \in V\backslash\{x_0\}$ such that $F(\bar{x}) \leq_{\operatorname{cl} C}^{(4)} F(x_0)$. From the condition (ii) in Lemma 3.6, it follows that $h_C(\bar{y};\hat{y_0}) \leq -h_C(-\hat{y_0};\hat{y_0})$. Hence we get $\psi_C^F(\bar{x};\hat{y_0}) \leq -\varphi_C^{-F}(x_0;\hat{y_0})$, which contradicts to the assumption.

On the other hand, suppose that x_0 is a strongly efficient solution with respect to $\leq_{\operatorname{cl} C}^{(4)}$. Then there no exists $x \in V\backslash\{x_0\}$ such that $F(x) \leq_{\operatorname{cl} C}^{(4)} F(x_0)$. Hence for any $x \in V\backslash\{x_0\}$ and $y_0 \in F(x_0)$, there exists $\hat{y} \in F(x)$ such that $y_0 \notin \hat{y} + \operatorname{cl} C$. From the condition (ii) in Lemma 3.5, it follows that $-h_C(-y_0;k) < h_C(\hat{y};k)$ for any $k \in \operatorname{int} C$. From the compactness of $F(x_0)$, there exists $\hat{y_0} \in F(x_0)$ such that $-h_C(-\hat{y_0};k) = \sup_{y_0 \in F(x_0)}\{-h_C(-y_0;k)\}$. Hence we get $-\varphi_C^{-F}(x_0;k) < \psi_C^F(x;k)$. □

Theorem 4.5. *Let $x_0 \in V$. Suppose $F(x_0) \subset \operatorname{int} C$ and $F(x)$ is compact for all $x \in V$. If x_0 is a strongly efficient solution for* (SVOP) *with respect to $\leq_{\operatorname{cl} C}^{(5)}$ then $-\varphi_C^{-F}(x_0;k) < \psi_C^F(x;k)$ for any $x \in V\backslash\{x_0\}$ and $k \in \operatorname{int} C$.*

Proof. Suppose that x_0 is a strongly efficient solution with respect to $\leq_{\operatorname{cl} C}^{(5)}$. Then there no exists $x \in V\backslash\{x_0\}$ such that $F(x) \leq_{\operatorname{cl} C}^{(5)} F(x_0)$. Hence for any $x \in V\backslash\{x_0\}$, there exists $\hat{y} \in F(x)$ such that $\hat{y} \notin y_0 - \operatorname{cl} C$ for any $y_0 \in F(x_0)$. From the condition (ii) in Lemma 3.5, it follows that $-h_C(-y_0;k) < h_C(\hat{y};k)$ for any $k \in \operatorname{int} C$. From the compactness of $F(x_0)$, there exists $\hat{y_0} \in F(x_0)$ such that $-h_C(-\hat{y_0};k) = \sup_{y_0 \in F(x_0)}\{-h_C(-y_0;k)\}$. Hence we get $-\varphi_C^{-F}(x_0;k) < \psi_C^F(x;k)$. □

Theorem 4.6. *Let $x_0 \in V$. Suppose $F(x_0) \subset \operatorname{int} C$ and $F(x)$ is compact for all $x \in V$. x_0 is a strongly efficient solution for* (SVOP) *with respect to $\leq_{\operatorname{cl} C}^{(6)}$ if and only if $-\varphi_C^{-F}(x_0;k) < \varphi_C^F(x;k)$ for any $x \in V\backslash\{x_0\}$ and $k \in \operatorname{int} C$.*

Proof. Suppose that $-\varphi_C^{-F}(x_0;k) < \varphi_C^F(x;k)$ for any $x \in V\backslash\{x_0\}$ and $k \in \operatorname{int} C$. Assume that x_0 in not a strongly efficient solution with respect to $\leq_{\mathrm{cl}\,C}^{(6)}$. Then there exists $\bar{x} \in V\backslash\{x_0\}$ such that $F(\bar{x}) \leq_{\mathrm{cl}\,C}^{(6)} F(x_0)$. From the condition (ii) in Lemma 3.6, it follows that $h_C(\hat{y};\hat{y}_0) \leq -h_C(-\hat{y}_0;\hat{y}_0)$. Hence we get $\varphi_C^F(\bar{x};\hat{y}_0) \leq -\varphi_C^{-F}(x_0;\hat{y}_0)$, which contradicts to the assumption.

On the other hand, suppose that x_0 is a strongly efficient solution with respect to $\leq_{\mathrm{cl}\,C}^{(6)}$. Then there no exists $x \in V\backslash\{x_0\}$ such that $F(x) \leq_{\mathrm{cl}\,C}^{(6)} F(x_0)$. Hence choose any $x \in V\backslash\{x_0\}$, then $y \notin y_0 - \mathrm{cl}\,C$ for any $y \in F(x)$ and $y_0 \in F(x_0)$. From the condition (ii) in Lemma 3.5, it follows that $-h_C(-y_0;k) < h_C(y;k)$ for any $k \in \operatorname{int} C$. The compactness of $F(x)$ and $F(x_0)$, there exist $\hat{y} \in F(x)$ and $\hat{y}_0 \in F(x_0)$ such that $h_C(\hat{y};k) = \inf_{y\in F(x)}\{h_C(y;k)\}$ and $-h_C(-\hat{y}_0;k) = \sup_{y_0\in F(x_0)}\{-h_C(-y_0;k)\}$, respectively. Therefore, $-\varphi_C^{-F}(x_0;k) < \varphi_C^F(x;k)$. □

5 Saddle Point Concepts for Set-Valued Maps

In the section, we stated an existence theorem for loose saddle point of set-valued maps by using inherited properties of convexity and semicontinuity of set-valued maps through the scalarizing method in Section 3. Besides, we state several notions for saddle point of set-valued maps, and propose a new idea on efficient-like saddle points in order to apply results given in Section 4.

At first, we give the preliminary terminology used throughout this section. Let X and Y be two Hausdorff topological vector spaces, and Z be a real ordered topological vector space with the vector ordering $\leq_C$ induced by a convex cone C in the same manner of Section 2. For C, an element x_0 of a subset A of Z is said to be a C-minimal point of A (or an efficient point of A with respect to C) if $\{x \in A \mid x \leq_C x_0, x \neq x_0\} = \emptyset$, which is equivalent to $A \cap (x_0 - C) = \{x_0\}$. We denote the set of all C-minimal points of A by Min$A$. Also, C^0-minimal [resp., C-maximal, C^0-maximal] set of A is defined similarly where $C^0 := \operatorname{int} C \cup \{0\}$, and denoted by $\mathrm{Min}_{\mathrm{w}}A$ [resp., Max$A$, $\mathrm{Max}_{\mathrm{w}}A$]. These C^0-minimality and C^0-maximality are weaker than C-minimality and C-maximality, respectively.

Let $A \subset X$ and $B \subset Y$, and $f : A \times B \to Z$ and $F : A \times B \to 2^Z$ be a vector-valued function and a set-valued map, respectively.

Definition 5.1. A point (x_0, y_0) is said to be with respect to $A \times B$:

(i) a C-saddle point of f if $f(x_0,y_0) \in \mathrm{Max} f(x_0,B) \cap \mathrm{Min} f(A,y_0)$;

(ii) a weak C-saddle point of f if $f(x_0,y_0) \in \mathrm{Max}_{\mathrm{w}} f(x_0,B) \cap \mathrm{Min}_{\mathrm{w}} f(A,y_0)$;

(iii) a C-saddle point of F if $F(x_0,y_0) \cap \mathrm{Max} F(x_0,B) \cap \mathrm{Min} F(A,y_0) \neq \emptyset$;

(iv) a weak C-saddle point of F if $F(x_0,y_0) \cap \mathrm{Max}_{\mathrm{w}} F(x_0,B) \cap \mathrm{Min}_{\mathrm{w}} F(A,y_0) \neq \emptyset$;

(v) a C-loose saddle point of F
if $F(x_0,y_0) \cap \mathrm{Max} F(x_0,B) \neq \emptyset$ and $F(x_0,y_0) \cap \mathrm{Min} F(A,y_0) \neq \emptyset$;

(vi) a weak C-loose saddle point of F
if $F(x_0, y_0) \cap \mathrm{Max_w} F(x_0, Y) \neq \emptyset$ and $F(x_0, y_0) \cap \mathrm{Min_w} F(X, y_0) \neq \emptyset$.

We note that any C-saddle point of f is a weak C-saddle point of f, and that any C-saddle (resp., C-loose saddle) point of F is a weak C-saddle (resp., weak C-loose saddle) point of F, obviously. Moreover, any C-saddle (resp., weak C-saddle) point of F becomes a C-loose saddle (resp., weak C-loose saddle) point of F. Also, in the case $C^0 \doteq C$, the conditions (i) and (ii) are coincident. We have three types of existence theorem of weak C-saddle points for vector-valued functions, and that of C-loose saddle point for set-valued maps.

Theorem 5.1. (cf. [1, 10].) *Let $A \subset X$ and $B \subset Y$ be two nonempty compact convex sets. If a set-valued map $F : A \times B \to 2^Z$ satisfies the following conditions:*

(i) *F is compact-valued and upper semicontinuous on $A \times B$ such that*
  (a) *for any $x \in A$, $F(x, \cdot)$ is C-lower semicontinuous on B,*
  (b) *for any $y \in B$, $F(\cdot, y)$ is $(-C)$-lower semicontinuous on A,*

(ii) *for any $x \in A$, $F(x, \cdot)$ is C-quasiconcave on B,*

(iii) *for any $y \in B$, $F(\cdot, y)$ is C-quasiconvex on A,*

then F has a weak C-loose saddle point with respect to $A \times B$.

Finally, we propose a new idea of saddle point concept for set-valued maps based on direct comparisons between sets given in Definition 2.1.

Definition 5.2. A point (x_0, y_0) is said to be an efficient saddle (resp., strongly efficient saddle) point of F on $A \times B$ with respect to $\leq_C^{(i)}$ for $i = 1, \ldots, 6$ if for any $(x, y) \in A \times B$, the following conditions hold for $i = 1, \ldots, 6$, respectively:

(i) $F(x, y_0) \leq_C^{(i)} F(x_0, y_0)$ implies $F(x_0, y_0) \leq_C^{(i)} F(x, y_0)$
(resp., $F(x, y_0) \leq_{\mathrm{cl}\,C}^{(i)} F(x_0, y_0)$ implies $F(x_0, y_0) \leq_{\mathrm{cl}\,C}^{(i)} F(x, y_0)$)

(ii) $F(x_0, y_0) \leq_C^{(i)} F(x_0, y)$ implies $F(x_0, y) \leq_C^{(i)} F(x_0, y_0)$
(resp., $F(x_0, y_0) \leq_{\mathrm{cl}\,C}^{(i)} F(x_0, y)$ implies $F(x_0, y) \leq_{\mathrm{cl}\,C}^{(i)} F(x_0, y_0)$)

We can verify each saddle point of F with Theorems 4.1–4.6.

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